Tema 3. Transformada de Laplace

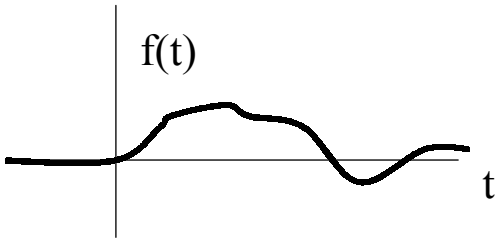
- *Concepto de transformada de Laplace.*
- *Concepto de transformada inversa de Laplace.*
- *Tablas de transformadas*
- *Propiedades de la transformada de Laplace*
- *Aplicaciones de la transformada de Laplace*

Concepto de transformada de Laplace

Dada una señal continua

f(t) función temporal para t≥0.

f(t) = 0 para t < 0



Se define su transformada de Laplace como :

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Siendo s la variable compleja de Laplace, $s = \sigma + j\omega$

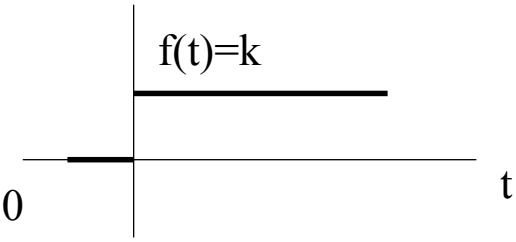
Cambio de variable t ⇒ s

La transformada de Laplace es a las funciones lo que el logaritmo a los números, es decir, una expresión alternativa con propiedades más convenientes

Ejemplo: función salto o escalón

$f(t) = 0$ para $t < 0$

$f(t) = k$ para $t >= 0$



$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty ke^{-st}dt = -k\frac{e^{-st}}{s}\Big|_0^\infty = \frac{k}{s}$$

Transformada inversa de Laplace

$$x(t) = L^{-1}[X(s)] = \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

Cambio de variable s $\Rightarrow$ t

Planteamiento del problema en el dominio temporal

$\xrightarrow{\text{Transformada de Laplace}}$

Resolución del problema en el dominio s $\Rightarrow$ X(s)

$\downarrow$ Transformada inversa de Laplace

Interpretación y expresión de la solución en el dominio t

Uso de tablas para calcular L(f(t)) y L^{-1}(F(s))

Tablas de transformadas de Laplace de las funciones más comunes

Función Temporal	*Transformada de Laplace*
$\delta(t)$	1
cte	$\frac{cte}{s}$
$e^{-a\cdot t}$	$\frac{1}{s+a}$
$t\cdot e^{-a\cdot t}$	$\frac{1}{(s+a)^2}$
$\cos(w\cdot t)$	$\frac{s}{s^2+w^2}$
$\mathrm{sen}(w\cdot t)$	$\frac{w}{s^2+w^2}$
$e^{-a\cdot t}\cdot\cos(w\cdot t)$	$\frac{s+a}{(s+a)^2+w^2}$
$e^{-a\cdot t}\cdot\mathrm{sen}(w\cdot t)$	$\frac{w}{(s+a)^2+w^2}$

Propiedades de la T. Laplace

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Linealidad $\longrightarrow$ $L[af(t)+bg(t)] = aF(s)+bG(s)$

Diferenciación $\longrightarrow$ $L\left[\frac{df(t)}{dt}\right] = sF(s)-f(0)$

$$L\left[\frac{d^2f(t)}{dt^2}\right] = s^2F(s)-s\cdot f(0)-\frac{df(0)}{dt}$$

Traslación en el tiempo $\longrightarrow$ $L[f(t-d)] = e^{-sd}F(s)$

Traslación compleja $\longrightarrow$ $L[e^{at}f(t)] = F(s-a)$

Derivada respecto a s $\longrightarrow$ $L[t\cdot f(t)] = -\frac{dF(s)}{ds}$

Cambio de escala $\longrightarrow$ $L\left[f(\frac{t}{a})\right] = a\cdot F(a\cdot s)$

Propiedades de la T. Laplace

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Valor final $\longrightarrow$ $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Valor inicial $\longrightarrow$ $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Integral de convolución $\longrightarrow$ $L\left[\int_0^\infty f(\tau)g(t-\tau)d\tau\right] =$

$$= L\left[\int_0^\infty g(\tau)f(t-\tau)d\tau\right] = F(s)G(s)$$

Propiedades I

$$\boxed{L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt}$$

$$L[af(t) + bg(t)] = aF(s) + bG(s)$$

$$L[af(t) + bg(t)] = \int_0^\infty [af(t) + bg(t)]e^{-st}dt = a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}dt = aF(s) + bG(s)$$

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0) \qquad L\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt$$

$$\int u\,dv = uv - \int v\,du \qquad dv = \frac{df(t)}{dt}dt \quad u = e^{-st} \quad \Rightarrow v = f(t) \quad du = -se^{-st}dt$$

$$L\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt = \left[e^{-st}f(t)\right]_0^\infty + \int_0^\infty f(t)se^{-st}dt = -f(0) + sF(s)$$

Propiedades II

$$L[f(t-d)] = e^{-sd}F(s)$$

$$L[f(t-d)] = \int_0^\infty f(t-d)e^{-st}dt \qquad t-d=\tau \quad t=0 \Rightarrow \tau=-d; \quad t=\infty \Rightarrow \tau=\infty$$

$$\int_0^\infty f(t-d)e^{-st}dt = \int_{-d}^\infty f(\tau)e^{-s(\tau+d)}d\tau = \int_0^\infty f(\tau)e^{-sd}e^{-s\tau}d\tau = e^{-sd}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-sd}F(s)$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad sF(s) = \int_0^\infty \frac{d\,f(t)}{d\,t}e^{-st}dt + f(0)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\int_0^\infty \frac{d\,f(t)}{d\,t}e^{-st}dt + f(0) = \int_0^\infty \frac{d\,f(t)}{d\,t}dt + f(0) =$$

$$= f(t)\Big|_0^\infty + f(0) = f(\infty) - f(0) + f(0) = f(\infty)$$

Propiedades III

$$L\left[\int_0^\infty f(\tau)g(t-\tau)d\tau\right] = F(s)G(s)$$

$$L\left[\int_0^\infty f(\tau)g(t-\tau)d\tau\right] = \int_0^\infty\left[\int_0^\infty f(\tau)g(t-\tau)d\tau\right]e^{-st}dt$$

$$t-\tau=\alpha \qquad t=0\Rightarrow\alpha=-\tau; \quad t=\infty\Rightarrow\alpha=\infty$$

$$\int_0^\infty\left[\int_0^\infty f(\tau)g(t-\tau)d\tau\right]e^{-st}dt = \int_0^\infty\left[\int_0^\infty f(\tau)g(t-\tau)e^{-st}d\tau\right]dt = \int_{-\tau}^\infty\left[\int_0^\infty f(\tau)g(\alpha)e^{-s(\alpha+\tau)}d\tau\right]d\alpha =$$

$$= \int_{-\tau}^\infty\left[\int_0^\infty f(\tau)e^{-s\tau}d\tau\right]g(\alpha)e^{-s\alpha}d\alpha = \left[\int_0^\infty f(\tau)e^{-s\tau}d\tau\right]\int_{-\tau}^\infty g(\alpha)e^{-s\alpha}d\alpha =$$

$$= \left[\int_0^\infty f(\tau)e^{-s\tau}d\tau\right]\int_0^\infty g(\alpha)e^{-s\alpha}d\alpha = F(s)G(s)$$

Aplicaciones: resolución de LODES

Ejemplo:

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=\frac{du}{dt}-0.5u \qquad y(0)=0; \quad \frac{dy(0)}{dt}=0; \quad u(t)=e^{-2t} \text{ para } t\geq 0$$

$$L\left[\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y\right]=L\left[\frac{du}{dt}-0.5u\right]$$

$$s^2Y(s)+2sY(s)+Y(s)=sU(s)-0.5U(s) \qquad Y(s)\left(s^2+2s+1\right)=(s-0.5)U(s)$$

$$Y(s)=\frac{s-0.5}{s^2+2s+1}U(s) \qquad U(s)=\frac{1}{s+2} \qquad Y(s)=\frac{s-0.5}{s^2+2s+1}\frac{1}{s+2}$$

$$y(t)=L^{-1}[Y(s)]=L^{-1}\left[\frac{s-0.5}{s^2+2s+1}\frac{1}{s+2}\right]=.....$$

$$dominio \quad t \quad \Rightarrow dominio \quad s \quad \Rightarrow dominio \quad t$$

Descomposición en fracciones simples

$$y(t)=L^{-1}[Y(s)]=L^{-1}\left[\frac{s-0.5}{s^2+2s+1}\frac{1}{s+2}\right]=L^{-1}\left[\frac{s-0.5}{(s+1)^2}\frac{1}{s+2}\right]$$

$$\frac{s-0.5}{(s+1)^2}\frac{1}{s+2}=\frac{a}{s+2}+\frac{b}{s+1}+\frac{c}{(s+1)^2}$$

$$\frac{s-0.5}{(s+1)^2}\frac{1}{s+2}=\frac{a(s+1)^2}{(s+1)^2(s+2)}+\frac{b(s+1)(s+2)}{(s+1)^2(s+2)}+\frac{c(s+2)}{(s+1)^2(s+2)}$$

$$s=-1 \quad \Rightarrow \quad -1.5=c$$

$$s=-2 \quad \Rightarrow \quad -2.5=a$$

$$s=0 \quad \Rightarrow \quad -0.5=a+2b+2c=-5.5+2b \quad \Rightarrow \quad b=2.5$$

$$y(t)=L^{-1}\left[\frac{-2.5}{s+2}+\frac{2.5}{s+1}+\frac{-1.5}{(s+1)^2}\right]=-2.5e^{-2t}+2.5e^{-t}-1.5te^{-t}$$